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Engineering Notes

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Optimum Loading of a Wing with a Slit

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Nomenclature

Æ = aspect ratio of wing
A_n = Fourier coefficient
b = span of the wing
C_{Di} = induced drag coefficient
C_L = lift coefficient
F = objective function
jx = maximum number of points on each span
U = incoming flow velocity
w = downwash (ref. U)
x, y, z = Cartesian coordinate system (ref. $b/4$)
Γ = circulation (ref. $Ub/4$)
λ = Lagrange multiplier

Problem Statement

THE conjecture that a wing with a very small slit in the plane of symmetry has an elliptic optimum loading is proved. The convergence toward the solution, however, is very slow with the number of terms in the Fourier-series expansion, as shown by the analysis and numerical solution.

Consider two lifting lines placed along the y axis, of equal length $b/2$, that barely touch the origin O (Fig. 1). The flow is assumed inviscid and incompressible, with incoming velocity U in the x direction.

Let $\Gamma(y)$ be the optimum loading. Γ must satisfy the following boundary conditions:

$$\Gamma(-b/2) = \Gamma(0) = \Gamma(b/2) \tag{1}$$

According to Prandtl lifting-line theory, the circulation can be expanded in Fourier series as

$$\begin{cases} \Gamma[y(t)] = 2Ub\sum_{n=1}^{\infty} A_n \sin nt \\ y(t) = -\dfrac{b}{2}\cos t, \qquad 0 \le t \le \pi \end{cases} \tag{2}$$

The induced drag is given by

$$C_{\mathrm{Di}}/\pi Æ = A_1^2 + 3A_3^2 + \cdots + (2p+1)A_{2p+1}^2 + \cdots \tag{3}$$

Only the odd terms have been retained for reason of symmetry.

The downwash is given by

$$w[y(t)] = -U\sum_{n=1}^{\infty} nA_n \frac{\sin nt}{\sin t} \tag{4}$$

The lift depends on the first mode amplitude only:

$$C_L/\pi Æ = A_1 \tag{5}$$

The conditions $\Gamma(-b/2) = \Gamma(b/2)$ are satisfied by the Fourier series of sines. The condition $\Gamma(0) = 0$, on the other hand, requires the coefficients to satisfy the constraint:

$$0 = \Gamma(0) = A_1 - A_3 + A_5 + \cdots + (-1)^p A_{2p+1} + \cdots \tag{6}$$

which states that the lifting-line tips, although very close to touching, have zero circulation.

Optimum Loading

Let

$$F = C_{\mathrm{Di}}/\pi Æ + \lambda\Gamma(0) \tag{7}$$

be the objective function and λ the Lagrange multiplier associated with the constraint.

The total lift being fixed, there is no variation of A_1, and the Euler–Lagrange system reads

$$\frac{\partial F}{\partial A_3} = 6A_3 - \lambda = 0, \qquad \frac{\partial F}{\partial A_5} = 10A_5 + \lambda = 0$$
$$\frac{\partial F}{\partial A_{2p+1}} = 2(2p+1)A_{2p+1} + (-1)^p\lambda = 0 \qquad p = 1, \ldots, \infty \tag{8}$$

This is an infinite system. Let us consider successive levels of approximation consisting of retaining the first $m+1$ coefficients only, that is, $A_1, A_3, A_5, \ldots, A_{2m+1}$ and setting the others to zero. Then, m will be allowed to increase to infinity to obtain the result. This is equivalent to searching for the optimum solution is the space spanned by the first $m+1$ Fourier modes. From the preceding system we get

$$A_{2p+1} = -\frac{(-1)^p}{2(2p+1)}\lambda \qquad p = 1, \ldots, m \tag{9}$$

The value of the Lagrange multiplier is obtained by satisfying the constraint with the first $m+1$ terms:

$$A_1 - A_3 + A_5 + \cdots + (-1)^m A_{2m+1}$$
$$= A_1 - \tfrac{1}{2}\left[\tfrac{1}{3} + \tfrac{1}{5} + \cdots + 1/(2m+1)\right]\lambda = 0 \tag{10}$$

Define the partial sum

$$S_m = \tfrac{1}{3} + \tfrac{1}{5} + \cdots + 1/(2m+1) \tag{11}$$

then, one finds

$$\lambda = 2(A_1/S_m) \tag{12}$$

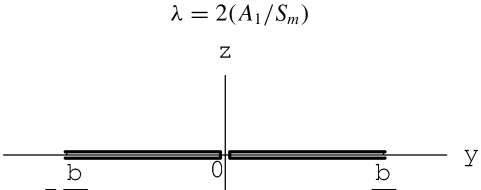

Fig. 1 Schematic of the problem.

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The coefficients now read

$$A_{2p+1} = -\left[(-1)^p \big/ (2p+1)S_m\right] A_1 \tag{13}$$

Note that, for large values of m, S_m behaves as $\frac{1}{2}\,\ell n(m)$, a diverging sum. Hence, for m sufficiently large, the coefficients A_{2p+1} can be made arbitrarily small in absolute value.

Let

$$C_{\mathrm{Dim}}/\pi AR = A_1^2 + 3\left(A_1^2 \big/ 3^2 S_m^2\right) + \cdots + (2m+1)\left[A_1^2 \big/ (2m+1)^2 S_m^2\right] \tag{14}$$

be the optimum induced drag using the first $m+1$ modes. The induced drag can be simplified to read

$$C_{\mathrm{Dim}}/\pi AR = A_1^2\left\{1 + \left(\tfrac{1}{3} + \tfrac{1}{5} + \cdots + 1/(2m+1)\right)\left(1\big/S_m^2\right)\right\} = A_1^2(1 + 1/S_m) \tag{15}$$

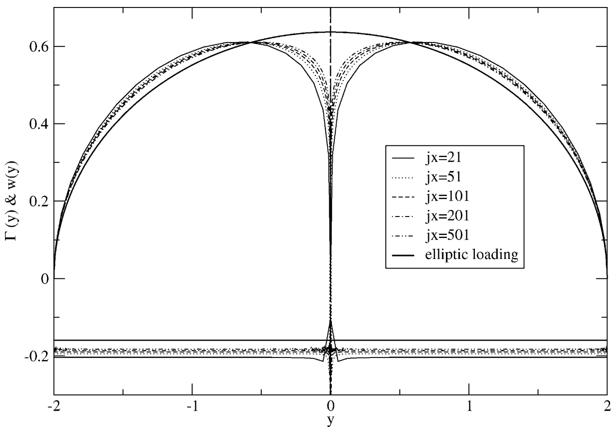

Fig. 2 Convergence of circulation with mesh refinement.

As $m \to \infty$, $(C_{\mathrm{Di}}/\pi AR)_m \to A_1^2$, which corresponds to the elliptic loading. All of the terms $A_3, A_5, \ldots, A_{2p+1}$ tend to zero.

The convergence with m to the elliptic distribution is very slow; however, as can be seen from the analysis and in the numerical solution, finer and finer meshes produce only small changes in the circulation (Fig. 2). The numerical solution is based on the minimization of the induced drag for a given lift.[1,2] The numerical solution corresponds to a biplane code where the two wings of span $b = 2$ are placed tip to tip and compared with a wing of span $b = 4$ with elliptic distribution. The two wings are meshed separately with cosine distributions that cluster the points near $y = -2$, 0, and 2, the mesh containing respectively $jx = 21, 51, 101, 201$, and 501 points on each lifting line. The circulation $\Gamma = 2\cos(t)/\pi$ corresponds to a lift coefficient of two, with a reference area $A_{\mathrm{ref}} = (b/4)^2$. The downwash for the elliptic loading is $w = -1/(2\pi)$. It would take of the order of 10^{86} Fourier coefficients to be within 1% accuracy of the induced drag of the elliptic wing.

Conclusion

It has been shown, both analytically and with a computer code, that the optimum loading of a wing with a slit in the plane of symmetry is elliptic. It would be interesting to prove that the same result holds for a slit located at an arbitrary position along the span.

Acknowledgment

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References

[1]Chattot, J. J., "Analysis and Design of Wings and Wing/Winglet Combinations at Low Speeds," AIAA Paper 2004-0220, Jan. 2004.

[2]Chattot, J. J., "Analysis and Design of Wings and Wing/Winglet Combinations at Low Speeds," *Computational Fluid Dynamics Journal*, Vol. 13, No. 3, Oct. 2004.